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Finding Lie groups that reduce the order of discrete dynamical systems

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Abstract. Discrete dynamical systems of the form $x_{m+1} = f(x_m)$ are considered, where x_m is an n-component vector. Equations $X = f(x)$ define a mapping f from an n-dimensional projective space into itself. Each component of f is a rational function, i.e. a ratio of polynomials in n dynamical variables. Maeda showed that when f commutes with each transformation g_α of a Lie group, a reduction in the order of the dynamical system results. Given a discrete dynamical system, the difficulty is to find its continuous symmetries. We present a way of using f-invariant sets to find these symmetries. The approach taken is to arrange groups in order of increasing complication and to characterize the set of dynamical systems admitting each group. Criteria are given for recognizing and reducing the order of systems admitting subgroups of the projective general linear group in n variables, PGL(n), or certain Lie subgroups of the Cremona group of birational transformations in n variables, $\mathcal{C}_n$. Quispel *et al* demonstrated the use of canonical group variables for achieving this reduction. We develop canonical coordinates for several groups with elementary Lie algebras and demonstrate reduction of order in each case. Results are used to reduce the order of several examples of recursion formulae taken from the literature on renormalizable lattice models.

1. Introduction

Lie theory includes all known solution methods for systems of differential equations. Each method amounts to a special case of Lie's procedure of finding continuous groups that leave the system invariant and using them to reduce the order [1]. The books by Cohen [2] and Stephani [3] are useful introductions. Details are given in several standard works [4–10]. Leznov and Saleliev treat Lie groups and Lie algebras in the context of continuous dynamical systems [11]. This paper is about application of Lie groups to discrete dynamical systems arising from physical problems.

Interest in discrete dynamics derives from several sources. One is that systems as simple as the discrete logistic equation exhibit complicated behaviours such as period doubling. One can apply symbolic dynamics and related tools. Another source of interest is that sometimes discrete approximations for differential equations can be solved directly as difference equations. Then one can study the way solutions of the difference equations approximate those of the original differential equations. To construct solutions of differential equations as limits one would like the approximating difference systems to be integrable [12, 13]. But it is interesting that a variety of different continuum limits can come from the same discrete system. Continuum limits relate to renormalization fixed points of the difference system.

Statistical mechanical models on lattices are another source of interest in discrete dynamics. For instance, the real-space renormalization method leads to systems of recursion

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relations (nonlinear difference equations or discrete dynamical systems). The dynamical systems can often be made approximately finite dimensional. In the special case of problems on finitely ramified lattices, the exact renormalization recursions actually are finite dimensional. For these cases (a) to solve the difference system exactly is to solve the underlying physical problem exactly, so (b) one can expect whatever hidden symmetries the original problem has to be present also in the dynamical system. This means there is both a motive for trying to simplify such systems and a reason to hope that a simplification of some kind can occur. In addition to the renormalizable fractals, an important set of models is solvable on regular lattices in two dimensions [14, 15]. A key to their solvability is the Yang–Baxter consistency relation. New solvable models can be generated by finding solutions to these consistency conditions on the Boltzmann weights, which in turn satisfy difference equations known to admit symmetries [16–18].

One can think of a dynamical system as a mapping of the space of dynamical variables into itself. We consider Lie groups that also consist of mappings of the space into itself. From this point of view the system and the group operations are on the same footing. However, each group transformation must have an inverse, while the system map generally does not. More over, the set of group operations is continuous with respect to the group parameters. Nevertheless, the fundamental symmetry criterion is given most easily in terms of mappings in the following way: When the system map commutes with each of the group transformations one says the system admits the group or is equivariant with respect to the group. The significance of this is that the group can then be used to reduce the order of the dynamical system.

Lie theory gives an algorithm for finding the largest group a system of differential equations admits (when the combined order is greater than one) [3]. The problem with extending the theory to difference equations is that no such algorithm is known for the discrete case. Several authors [19–24] have developed methods of detecting the symmetry groups of the dynamical system. Chossat and Golubitsky [20, 21] have studied the problem in connection with bifurcation and the symmetry of attractors. They prove a general theorem on the connection between invariant sets and a subgroup of $O(n)$ which is a symmetry group of the dynamical system. In particular they show that the system maps the fixed point space of the group into itself and they also prove a theorem on the connection between period two points and the associated symmetry group. Dellnitz *et al* [22, 24] prove several results about the transitivity of invariants sets and develop a method of detecting symmetries based on the Karhunen–Loève decomposition.

Concentrating on integrable systems and developing a theory in terms of complete sets of invariant functions, Maeda [12] gave criteria for determining when a given system of difference equations admits a given group. He also extended the idea of prolongation of the group generators to the difference case and obtained a functional equation for generators of the symmetry group of a given dynamical system. The analysis is restricted to symmetries that do not involve the discrete independent variable. Quispel *et al* [25] demonstrated the use of canonical group coordinates for reducing the order of a difference equation one step at a time. The treatment includes symmetries for which the independent variable appears in the transformation formulae. It was pointed out that even autonomous difference systems can admit such symmetries. They showed that sometimes the functional equation for the generators can be integrated by expanding both the difference equation and the generators at a fixed point. This permits solving term by term for a series representation of the generators. Unfortunately one must surmise the form of the general term, sum the series, and integrate to obtain the canonical coordinates in order to reduce the order of the original difference equation. Byrnes *et al* [26] showed that a difference equation with explicit dependence on the independent variable can be transformed into a linear one with constant coefficients if and only if it admits a symmetry that factors in a certain way.

Recently [27–29] one of the authors (WS) with Giona and Schwalm applied a group theoretic analysis to discrete dynamical systems that come from renormalizing finite-difference wave equations and transport equations on regular fractals. The strategy for finding groups rests on certain relationships between fixed or invariant sets of the renormalization recursions (i.e. of the dynamical system) and the fixed or invariant sets of any group they admit. By fixed set of a given map we mean a set of points each of which remains fixed, while an invariant set is a set mapped into itself.

Consider a discrete physical model consisting of a quadrature grid and a system of coupled ODEs, one for each lattice site or grid point, for which time is the continuous independent variable. By Laplace or Fourier transforming with respect to time, one arrives at a difference system corresponding roughly to the Helmholtz PDE. Difference models of wave mechanics or transport, such as the discrete Schrödinger equation or diffusion equation, are related to one another by changes of time dependence and boundary conditions. For a class of such models involving discrete Laplacian operators, it has been shown [28] that the process of finding continuous symmetries simplifies greatly when the problem is mapped onto diffusion with closed (zero flux) boundary conditions. The simplification is related to the existence of conserved currents which take on their simplest form for the diffusion problem with closed boundaries [28, 30].

This paper gives technical details necessary for reducing the order of discrete dynamical system $\mathcal{S}$ using the group $\mathcal{G}$ of its continuous symmetries. Section 2 defines the dynamical systems of interest. Section 3 summarizes basic notation for connected Lie groups. Correspondence of the fixed and invariant sets of $\mathcal{S}$ with those of $\mathcal{G}$ which $\mathcal{S}$ admits is described in section 4. Sections 5 and 6 give a systematic development for $\mathcal{S}$ admitting subgroups of $\mathrm{PGL}(n)$ and certain subgroups of the Cremona group $\mathcal{C}_n$ respectively. These sections address the problems of determining whether a given system admits such a group, and if so, how to match the group parametrization to dynamical properties of $\mathcal{S}$. A compilation of canonical variables for groups with several elementary types of Lie algebras is given in section 7. It is shown that introduction of canonical variables leads to reduction of order of $\mathcal{S}$ in each case. The method is illustrated in section 8 where it is used to decouple a collection of examples, most of which come from the literature on lattice models.

2. Discrete dynamical system $\mathcal{S}$

Following Maeda [12], we consider a discrete dynamical system $\mathcal{S}$ as a mapping of an n-dimensional complex space into itself, $f : \mathbb{C}^n \to \mathbb{C}^n$, such that

$$X = f(x) \tag{2.1}$$

where x and X are n-component vectors. One thinks of a system of first order, autonomous difference equations. In the following, attention will be on systems for which the functions defining f are rational. That is to say, each component f^i is a ratio of polynomials in n variables. We do not require f to have a single valued inverse. Thus in general one cannot solve for x in terms of rational functions of X. Cases of non-trivial rational transformations with rational inverses (Cremona transformations) do exist [31, 32]. Dynamical properties of such transformations have been studied by Maillard *et al* [33–38] and Veselov [39].

In contrast to the one-variable case, there may be points where a rational function of several variables is undefined. In other words, in addition to regular points and points that map to infinity (which present no difficulty) there can be points in the dynamical space where the action of f is not defined. These are points where the numerator and denominator polynomials of at least one component f^i are both zero and at which no limit exists. Such points are tacitly excluded from some of the discussion. So is the closure of the union of the set of pth inverse images under f of such points for all p. One can restrict the domain of f to the open set of regular points that remains. However, in other cases the points where f is undefined play important roles in the development. This is true for systems admitting the Cremona subgroups defined in section 6. Not withstanding possible difficulties, the following analysis reduces the order of many discrete dynamical systems.

3. The Lie group $\mathcal{G}$ admitted by $\mathcal{S}$, generators and the Lie algebra

For our purposes, a connected Lie group $\mathcal{G}$ is a set of mappings of the dynamical space onto itself. A particular group element $g_\alpha : \mathbb{C}^n \to \mathbb{C}^n$ is indexed by an r-dimensional parameter α with components α^i for $1 \leqslant i \leqslant r$ in such a way that $\mathcal{G}$ is a group under composition, $g_\alpha\, x = x_\alpha$ is a continuous function of α

$$x_\alpha = u(x, \alpha) \tag{3.1}$$

and g_0 is the identity so that $x = u(x, 0)$.

For the case of one parameter, $\mathcal{G}$ is Abelian so the parametrization can be chosen as either $g_\alpha\, g_\beta = g_{\alpha+\beta}$ with real α and β, or $g_\alpha\, g_\beta = g_{\alpha\beta}$ with α and β real and non negative. Except where indicated we assume additive parametrization, but sometimes the multiplicative form results in simpler expressions. For multiplicative parameters the identity is $\alpha = 1$ rather than 0.

Let dynamical system $\mathcal{S}$ be defined by the recursion equation (2.1). A solution is a sequence $\{x_m\}$ of points in $\mathbb{C}^n$ such that

$$x_{m+1} = f(x_m). \tag{3.2}$$

Following Lie [1] and Maeda [12], we say that $\mathcal{G}$ is a symmetry group of $\mathcal{S}$ or that $\mathcal{S}$ admits $\mathcal{G}$ if $\mathcal{G}$ takes solutions into solutions, so that whenever equation (3.2) holds

$$u(x_{m+1}, \alpha) = f(u(x_m, \alpha)) \tag{3.3}$$

or

$$u(f(x), \alpha) = f(u(x, \alpha)). \tag{3.4}$$

It is equivalent [12] to say that the mappings f and g_α commute for each α, so that

$$f\, g_\alpha = g_\alpha\, f. \tag{3.5}$$

An alternative terminology for the commutivity of f with group members g_α is to say that the map f is equivariant with the group $\mathcal{G}$ (see [19]). Expressed graphically the condition is that the following diagram commutes for each g_α.

$$\begin{array}{ccc} x & \xrightarrow{f} & X \\ g_\alpha \downarrow & & \downarrow g_\alpha \\ x_\alpha & \xrightarrow{f} & X_\alpha. \end{array}$$

We call any map such as g_α in equation (3.5) that commutes with f a symmetry of $\mathcal{S}$.

Quispel *et al* [25] point out that although the system $\mathcal{S}$ may be autonomous, the most general symmetry can involve the discrete, independent variable m in equation (3.2). The group does not transform m, but x transforms in a way that depends on m as well as on α so that $x_\alpha = u(x, m, \alpha)$ and $X_\alpha = u(X, m+1, \alpha)$. The general criterion for a non-autonomous dynamical system to admit such a group [25] is

$$u(f(x, m), m + 1, \alpha) = f(u(x, m, \alpha), m). \tag{3.6}$$

We will use group generators to obtain canonical coordinates that decouple the dynamical system. For each parameter α^i, a generator is defined by power series expansion near the identity

$$u(x, \alpha) = x + \alpha^i \xi_i(x) + \cdots \tag{3.7}$$

with summation on the repeated index. The components of the tangent vector $\xi_i(x)$ corresponding to α^i are,

$$\xi_i^j(x) = \left.\frac{\partial u^j(x, \alpha)}{\partial \alpha^i}\right|_{\alpha=(0,0,\ldots)}. \tag{3.8}$$

Tangent vector $\xi_i(x)$ for fixed i corresponds to the group generator $\mathcal{L}_i$ which is the directional derivative

$$\mathcal{L}_i = \xi_i^j(x) \frac{\partial}{\partial x^j}. \tag{3.9}$$

Then with $\alpha \cdot \mathcal{L} = \alpha^i \mathcal{L}_i$, since $\mathcal{G}$ is connected, one has

$$\mathrm{e}^{\alpha\cdot\mathcal{L}} x = u(x, \alpha) \tag{3.10}$$

so that $g_\alpha = \mathrm{e}^{\alpha\cdot\mathcal{L}}$. The left-hand side of equation (3.10) is the Taylor series expansion with respect to α of the functions $u(x, \alpha)$ which define g_α.

The Lie algebra L associated with the group $\mathcal{G}$ is generated by the set $\{\mathcal{L}_i\}_{i=1}^r$ with bracket $[\mathcal{L}_i, \mathcal{L}_j]$ denoting the commutator of directional derivatives. The linearity, anti-symmetry, and Jacobi non-associativity therefore follow immediately. The structure constants $\{C_{ij}^k\}$ appearing in

$$[\mathcal{L}_i, \mathcal{L}_j] = C_{ij}^k \mathcal{L}_k \tag{3.11}$$

characterize the Lie algebra completely [40]. A transformation of the coordinates x leaves the structure constants unchanged, while a change of parametrization α gives a linear transformation via a constant, non singular matrix. Thus reparametrizing can simplify the structure constants and put the algebra into a normal form [3].

4. Relation between fixed and invariant sets of $\mathcal{G}$ and of $\mathcal{S}$

For finding symmetries it is useful to know the relation between fixed or invariant sets of $\mathcal{S}$ and fixed or invariant sets of its symmetry group [27–29]. The first step is to find the fixed points from

$$f(x) - x = 0. \tag{4.1}$$

(Points at infinity in the sense of projective geometry are included using homogeneous coordinates as described below.) Some fixed points are isolated while others belong to continuously connected sets of fixed points. We shall see presently that isolated fixed points of $\mathcal{S}$ must also be fixed points of $\mathcal{G}$. This fact gives clues to the form that a symmetry group must take. Invariant sets of $\mathcal{S}$ are equally important, although we know of no systematic way to find them.

The set of pth inverse images of an isolated fixed point is another significant attribute of $\mathcal{S}$. Generally the map f defining $\mathcal{S}$ is not invertible. The inverse image or preimage $f^{-1}(x_0)$ of point x_0 is the solution set of

$$f(x) - x_0 = 0. \tag{4.2}$$

The preimage of a set is the union of the inverse images of its points. The pth inverse image of fixed point x_* is the set of all x which f^p takes into x_*.

Factoring equation (4.2) may resolve separate components of the inverse image. These are the sets upon which the individual factors vanish. One must be careful since computer algebra normally factors over the rationals, while physical problems pertain more naturally to the field of real or complex numbers. For example

$$x^4 - 2x^2y^2 + y^4 - 10x^2 - 2y^2 + 1 = (x^2 - y^2 - 2\sqrt{3}\,x + 1)\,(x^2 - y^2 + 2\sqrt{3}\,x + 1) \tag{4.3}$$

the zero set of which is the union of two hyperbolae, but the left-hand side is irreducible over the rationals. Typically in two dimensions, the zeros of the individual factors define curves. In higher dimensions the factors give higher dimensional sets.

Now consider the action of $\mathcal{G}$ on a fixed point of $\mathcal{S}$. Suppose $\mathcal{S}$ contains fixed point x_* so that

$$f(x_*) = x_*. \tag{4.4}$$

Application of g_α to x_* yields

$$u(x_*, \alpha) = x_{*\alpha}. \tag{4.5}$$

However, since g_α and system map f commute

$$u(f(x_*), \alpha) = f(u(x_*, \alpha)) \tag{4.6}$$

or combining equations (4.4)–(4.6) inclusive,

$$u(x_*, \alpha) = f(x_{*\alpha}). \tag{4.7}$$

Thus

$$x_{*\alpha} = f(x_{*\alpha}) \tag{4.8}$$

which shows that $x_{*\alpha}$ is also a fixed point. Therefore a symmetry must take each fixed point of $\mathcal{S}$ into either some other fixed point, or possibly itself. There are two cases: either x_* is isolated, meaning there is a neighbourhood of x_* containing no other fixed point, or else it is not.

When x_* is isolated it must also be a fixed point of the group, meaning $x_{*\alpha} = x_*$ for all α. One can see this in the following way. The image $x_{*\alpha}$ must be a continuous function of α; therefore we must have $\lim_{\alpha\to 0} x_{*\alpha} = x_*$. Assume by way of contradiction that $x_{*\alpha'} \neq x_*$ for some α'. In view of equation (4.8), choosing ϵ sufficiently small, we can find a fixed point of the form $x_{*,\epsilon\alpha'}$ in any deleted neighbourhood of x_* which contradicts the assumption that x_* is isolated. Thus when x_* is an isolated fixed point of $\mathcal{S}$ it is also a fixed point of $\mathcal{G}$. Or, in the terminology of [19], the isotropy subgroup of x_* is the whole group $\mathcal{G}$.

A by-product of the preceding argument is that when x_* is not isolated it belongs to a continuously connected set of fixed points which we denote as $\mathcal{G}\,x_*$. Evidently each element g_α of $\mathcal{G}$ permutes the points of $\mathcal{G}\,x_*$, i.e. it maps $\mathcal{G}\,x_*$ onto itself in one-to-one fashion. In fact fixed points of $\mathcal{S}$ are partitioned into equivalence classes modulo $\mathcal{G}$, each of which is either a single fixed point or a continuously connected set. These equivalence classes are the fixed-point manifolds of $\mathcal{S}$ discussed by Golubitsky *et al* [19].

A similar argument applies to cycles of $\mathcal{S}$. Suppose $\{x_1, x_2\}$ is a two-cycle, so that $f(x_1) = x_2$ and $f(x_2) = x_1$. Both x_1 and x_2 are fixed points of f^2, meaning that each satisfies $f(f(x)) = x$. Either they are isolated fixed points of f^2 or not. If so, then since f^2 commutes with g_α if f does, both x_1 and x_2 are fixed points of the group $\mathcal{G}$. If not, then each belongs to one of two connected sets, x_1 to one set and x_2 to the other. The system map f takes the sets onto one another in such a way that each point in one set is paired with one point in the other set to form a two-cycle of f. Thus the dynamical system exchanges the sets, while the group permutes each set. In general, a p-cycle of $\mathcal{S}$ is either isolated, in which case each of its members is a group fixed point, or else it belongs to continuously connected sets of p-cycles. This fact is useful for showing that a particular system does not admit groups of a given type. Roughly, one may find that a group $\mathcal{G}$ does not have sufficiently many fixed points to cover the isolated cycle points of $\mathcal{S}$.

An equally important result can be obtained for inverse images of fixed points. Recall that the inverse image $f^{-1}(x_*)$ is the set of points which map under a single application of f into x_*. Assuming point x is in the inverse image of x_* (isolated) equation (3.4) implies that

$$f(u(x,\alpha)) = u(x_*,\alpha) = x_*$$

where the result regarding isolated fixed points is used to obtain the final equality. Therefore, a symmetry maps $f^{-1}(x_*)$ into itself. Again, because the group must include the inverse $g_\alpha^{-1} = g_{-\alpha}$, the mapping g_α is one-to-one, i.e. g_α permutes the inverse image of an isolated fixed point.

If $f^{-1}(x_*)$ contains a one-dimensional set of points (an isolated curve) C parametrized by λ, and if f commutes with a one parameter subgroup ($r = 1$), then

$$f(u(x(\lambda),\alpha)) = u(f(x(\lambda)),\alpha) \tag{4.9}$$

implies that

$$f(u(x(\lambda),\alpha)) = u(x_*,\alpha) = x_* \tag{4.10}$$

assuming once again that x_* is an isolated fixed point. This means $u(x(\lambda),\alpha) \in f^{-1}(x_*)$ for all α and λ, as was shown previously (i.e. by the fact that $x(\lambda)$ is in the inverse image). However, $u(x(\lambda),\alpha)$ must tend to $x(\lambda)$ continuously as α goes to zero. Since C is an isolated curve, $u(x(\lambda),\alpha)$ is also on C. A group trajectory is a curve, $\{u(x_0,\alpha) : \alpha \in \mathbb{R}\}$, along which the image of an initial point moves as the parameter α varies. Thus the group moves initial points belonging to C along the curve, which therefore consists of trajectories of the group. These trajectories form segments of C that can be connected to one another only at fixed points of the group that are approached as $\alpha \to \pm\infty$.

If one trajectory is known then another may, in general, be obtained by applying f. Suppose a group trajectory C parametrized by λ is known such that for each α

$$u(x(\lambda),\alpha) = x(\lambda') \qquad \text{for } \lambda,\ \lambda' \in \mathcal{I} \tag{4.11}$$

for all λ where $\mathcal{I}$ is some interval used for parametrizing. Suppose f takes the curve C onto some other curve D defined as

$$D = \{\tilde{x}(\delta) : \delta \in \mathcal{I}\} = \{f(x(\lambda)) : \lambda \in \mathcal{I}\} \tag{4.12}$$

the image of the trajectory under the system mapping. Choosing, any δ_1

$$\begin{aligned} u(\tilde{x}(\delta_1), \alpha) &= u(f(x(\lambda_1)), \alpha) \\ &= f(u(x(\lambda_1), \alpha)) \\ &= f(x(\lambda_2)) \\ &= \tilde{x}(\delta_2) \end{aligned}$$

for some λ_1, λ_2, and δ_2 which satisfy the above conditions. Thus, D is another trajectory of the group since

$$u(\tilde{x}(\delta_1), \alpha) = \tilde{x}(\delta_2) \qquad \text{for } \delta_1, \delta_2 \in \mathcal{I}. \tag{4.13}$$

Finally, a one-dimensional inverse image of a group trajectory must be a set of group trajectories. Suppose that y is an arbitrary point element of the inverse image of x, so that $x = f(y)$. Then $u(x, \alpha) = u(f(y), \alpha) = f(u(y, \alpha))$ so that y belongs to a trajectory in the inverse image of the trajectory to which x belongs.

Thus we have demonstrated several relationships between fixed and invariant sets of a dynamical system and fixed and invariant sets of any group the system admits. The strategy given below for finding groups will use these relationships.

Knowing the structure of continuously connected sets of fixed points of $\mathcal{S}$ is also valuable for determining $\mathcal{G}$. As seen above, for fixed sets of higher dimension, e.g., surfaces, the only thing required of a symmetry is that it map the set back into itself in one-to-one fashion. Fixed sets of $\mathcal{S}$ that form one-dimensional curves are especially important. In particular, a fixed curve of $\mathcal{S}$ must be a trajectory of the symmetry group or must form a set of trajectories, the end points of which join at fixed points of $\mathcal{G}$. Inverse images of these trajectories under the system map f also consist of group trajectories. An intersection of trajectories must be a fixed point of $\mathcal{G}$.

Starting from the set of fixed points of $\mathcal{S}$ one takes inverse images to obtain invariant sets of $\mathcal{G}$. In two dimensions, factorization can often identify trajectories. In higher dimensions the process is more apt to result in higher dimensional invariant sets. However, such sets often occur in intersecting families. The one-dimensional subsets obtained as intersections of invariant sets comprise trajectories of the symmetry group. Once some group trajectories are determined, others can be found by continuing to take inverse images. The inverse image sets may or may not be one-dimensional curves. Sets that form curves must yield trajectories of $\mathcal{G}$.

Information obtained from the construction described in this section can yield pictures of the group flow detailed enough to reconstruct the group, e.g., by determining the form of its generators. But most importantly, this information is often sufficient to tell whether or not a given system $\mathcal{S}$ admits any of the elementary Lie groups classified in the following sections.

5. Systems admitting subgroups of PGL(n)

One approach taken by Lie to the problem of finding symmetries is to find the set of differential equations admitting a given group. Thus one can catalogue systems of equations systematically in order of increasing complexity of their symmetry groups. In this and the following section we treat discrete dynamical systems admitting subgroups of the projective general linear groups PGL(n) and certain subgroups of the quadratic Cremona groups in n variables respectively. The main tool is the collection of results in section 4 relating fixed and invariant sets of a group $\mathcal{G}$ with those of a dynamical systems $\mathcal{S}$ admitting $\mathcal{G}$. Finally by comparison with standard forms of PGL(n) and the Cremona subgroups we reduce the order of several examples of systems $\mathcal{S}$ appearing in the literature. It should be mentioned, though it may be obvious, that the dynamical system $\mathcal{S}$ itself is not supposed to consist of either a linear fractional or a Cremona transformation, although the latter is a special case of interest [33–39].

Discussion of PGL(n) [41] is simplified by the use of homogeneous coordinates. The n-dimensional projective space includes finite points x together with points at infinity. Let v be a non-zero vector of $n+1$ homogeneous coordinates such that $x_k = v_k/v_{n+1}$ for each component $1 \leqslant k \leqslant n$. Points at infinity in the projective space are the points for which $v_{n+1} = 0$. Points at infinity with homogeneous coordinate vectors $+v$ and $-v$ are identified as the same point. Therefore element g of PGL(n) corresponds to an invertible, linear transformation A acting on the homogeneous coordinates. There are $(n+1)^2$ entries in A but because a point x corresponds to a ray λv the group has $n(n+2)$ parameters. Fixed points correspond to eigenvectors of A.

Putting A in Jordan form C by means of a similarity is the same as putting g in a standard form $\hat{g}$ by conjugating with a fixed projective transformation γ. Thus the matrix $C = M\,A\,M^{-1}$ corresponds to the composition $\gamma\,g\,\gamma^{-1}$ which defines $\hat{g}$. One can classify a subgroup $\mathcal{G}$ of PGL(n) according to the form C and the transformation matrix M which puts the generalized eigenvectors of A in a standard configuration, i.e. along the coordinate axes.

The number of generalized eigenvectors of A is $n+1$, the dimension of the homogeneous space. The matrix M should take each generalized eigenvector into one of the coordinate basis vectors. We refer to the coordinates with respect to the basis of generalized eigenvectors of A as standard variables. They are related to the canonical group coordinates introduced in section 7. Different cases can be distinguished according to the Jordan form C.

Consider a two-dimensional system, $n = 2$. In the standard coordinates C is one of the following forms:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.1}$$

To each of these belongs a family of subgroups, characterized by properties of its fixed and invariant sets [41]. For simplicity, let the Jordan forms relate to homogeneous standard coordinates (u, v, w) and let $(x = u/w,\ y = v/w)$ be standard dynamical variables.

The first form in the list (5.1) has three distinct eigenvalues and three eigenvectors forming a basis where C is diagonal with eigenvalues a, b, and 1. Assume first that these are real. For this case, there are additive parameters α and β such that $\mathrm{e}^{\alpha} = a$ and $\mathrm{e}^{\beta} = b$ and g maps (x, y) to $(\mathrm{e}^{\alpha}\,x, \mathrm{e}^{\beta}\,y)$, so that the fixed points in standard coordinates are $(0, 0)$, $(\infty, 0)$ and $(0, \infty)$. One is a source, one a sink and the other a saddle, in the general case. In the special case when two of $\{a, b, 1\}$ are equal, there is a one parameter subgroup with a line of fixed points either on an axis or at infinity. The latter is a uniform dilation in the standard variables, which is of practical importance. The case of a symmetry that is a rotation or spiral group in the standard variables (which appears to be less common in physical problems) is included when a and b are complex conjugate. When the dynamical system $\mathcal{S}$ is complex, the eigenvalues and generalized eigenvectors are naturally complex.

For the second Jordan form, there are again two parameters $\mathrm{e}^{\alpha} = a$ and $\beta = b$, and (x, y) maps to $(\mathrm{e}^{\alpha}\,x + \beta\,y, \mathrm{e}^{\alpha}\,y)$. By permuting the standard homogeneous coordinates one also has $(\mathrm{e}^{\alpha}\,x, y + \beta)$ and $(x/(\beta\,x + 1), \mathrm{e}^{\alpha}\,y/(\beta\,x + 1))$ and so on. The subgroups of this type are characterized by a fixed point with an invariant line through it and a second distinct fixed point not on the invariant line. The special case $a = 1$ includes translations such as $(x, y + \beta)$. The feature characterizing this special case is a single line of fixed points.

The third form maps (x, y) to $(x + \alpha y, y + \alpha)$ with the additive group parameter $\alpha = a$. Again the coordinates may be permuted, but for each permutation the subgroup is still characterized by a single fixed point with a invariant line through it.

The question is whether or not a particular $\mathcal{S}$ admits a subgroup of $\mathrm{PGL}(n)$. If it does, then in the homogeneous coordinates v one can find a transformation that puts the matrix A of each element of a one-parameter subgroup in the same Jordan form C. Then it must be possible to express each entry of C in terms of the group parameter so as to satisfy the symmetry condition, equation (3.5).

Both the number and type of fixed and invariant sets of a dynamical system $\mathcal{S}$ admitting a subgroup $\mathcal{G}$ of $\mathrm{PGL}(n)$ are restricted. For the diagonalizable case of A with distinct eigenvalues, corresponding to the first Jordan form of equation (5.1), the number $n + 1$ of group fixed points limits the number of isolated fixed or cycle points of $\mathcal{S}$. When $\mathcal{G}$ fixes a subspace, $\mathcal{S}$ can have rather general sets of fixed points and cycles within the subspace, which must be a hyperplane of dimension $\leqslant n - 1$. When $\mathcal{G}$ has invariant hyperplanes, these must be either fixed or invariant hyperplanes of $\mathcal{S}$.

Therefore, given $\mathcal{S}$, one first finds the set of fixed points. If either the cardinality or the type does not fit the constraints of $\mathrm{PGL}(n)$, then the symmetry group $\mathcal{G}$ contains no subgroup of $\mathrm{PGL}(n)$. Next one looks at the set of p-cycles up to a convenient maximum p to see whether the cardinality or type rules out $\mathrm{PGL}(n)$. If not, one can try the assumption that there is indeed a projective linear symmetry. A single projective map should place the fixed points, cycle points and invariant sets of $\mathcal{S}$ in standard position. If the system admits a subgroup corresponding to one of the Jordan forms, the next step is to relate entries of C to group parameters in such a way as to form a symmetry of $\mathcal{S}$. In many cases this can be done by inspection, otherwise the problem reduces to one of dimensional analysis, since for each one-parameter subgroup, each matrix entry must be a power of the multiplicative group parameter. Each matrix element can therefore be set to a power of the parameter and the exponents treated as undetermined coefficients. The procedure is applied to physical examples in section 8 below.

6. Lie subgroups of $\mathcal{C}_n$

The Cremona group [31, 32] $\mathcal{C}_n$ is the group of all birational mappings taking the projective space of n variables into itself. To say a mapping is birational is to say that it is defined by rational functions, that it has an inverse, and that the inverse is also defined by rational functions.

For example, $\mathcal{C}_1$, the group of birational maps of the projective line into itself, is nothing other than PGL(1), namely the linear fractional or Möbius transformations. The group $\mathcal{C}_2$ is generated by PGL(2) and the single quadratic transformation taking (x, y) to $(1/x, 1/y)$, or in homogeneous coordinates, taking (u, v, w) to (vw, wu, uv). Knowledge of $\mathcal{C}_n$ for $n \geqslant 3$ is limited mostly to a collection of examples [32]. Apparently these include $\mathrm{PGL}(n)$ and the special conformal group in n variables as finitely generated subgroups.

Maillard *et al* [33, 34, 36–38] and Veselov [39] have studied dynamical systems defined by several types of Cremona mappings. These include polynomial maps in two dimensions that have polynomial inverses. Maillard and coworkers derive many interesting properties for dynamics of Cremona maps obtained as the products of involutions [35]. The dynamical variables are entries in a $q \times q$ matrix. The involutions are matrix inversion, diadic (element wise) inversion, transposition of certain matrix entries, etc. The maps have their origin in the theory of exactly solvable vertex models [14, 15].

Integrability of the system $\mathcal{S}$ depends on the way certain numerical characteristics (such as the complexity introduced by Arnold [42]) of the pth iteration f^p of the system map f grow with p. Veselov established this for the special case of polynomial automorphisms in (x, y). The relation between growth of complexity and integrability of the maps studied by Maillard *et al* was demonstrated for particular cases by Viallet and Falqui [43]. Our concern will be not with the dynamics of Cremona maps but with the larger class of dynamical systems which admit connected Lie subgroups of $\mathcal{C}_n$ as symmetries. We are unable to deal with the most general case. We restrict attention to certain quadratic Cremona subgroups. Nevertheless, these special subgroups are sufficient for decoupling a variety of maps that arise in physical problems, as will be seen in section 7.

Consider first the projective plane, $n = 2$. A quadratic Cremona transformation is one for which the degrees of the numerator and denominator polynomials defining the map in homogeneous coordinates are 2. The theorem of Bezout shows that the curves defined as zero sets of two polynomials of degrees m and n and having no common factors intersect at $m\,n$ points. One must include complex points, count degenerate intersections appropriately, and include points at infinity in the projective sense. Therefore two distinct quadratic curves intersect at four points.

Suppose $S_1(x, y)$ and $S_2(x, y)$ are quadratic polynomials in x and y, meaning that the highest combined power is 2 in each case. A pencil of quadratic curves, or conic sections, can be parametrized as $S_1(x, y) - aS_2(x, y) = 0$. Each curve in the family corresponds to a choice of a, and for any a the curve passes through the four points where the curves $S_1(x, y) = 0$ and $S_2(x, y) = 0$ intersect. We shall call these four points base points. Special cases arise when two base points coincide. For example, if $S_1(x, y) = y - (1 + x)^2$ and $S_2(x, y) = 2x^2$, there are two, distinct, complex base points and a confluence of two base points at ∞ along the y axis. At the confluence, all the curves of the family must have the same tangent. In general we assume the base points are distinct and that one can treat confluent base points as limits.

A general quadratic polynomial $P(x, y)$ has five parameters. Forcing the curve $P(x, y) = 0$ to pass through four base points fixes four of the coefficients, leaving only one free parameter b, say. Thus there is a unique one-parameter family of quadratic curves passing through the four points. Hence for a given set of base points, the parameter a in $S_1(x, y) - aS_2(x, y)$ must be an invertible function of the parameter b in $P(x, y)$ so that these two functions represent different parametrizations of the same family of curves. Thus it is convenient to choose the following parametrization. Let the lines $L_k(x, y) = 0$ for $k = 1, 2, 3$ or 4 each pass through two base points so as to form a quadrilateral. Let $P(x, y) = L_1L_2 - bL_3L_4$. According to Bezout's theorem, any line must intersect each curve in the family exactly twice. Let $Q(x, y) = L_1 - aL_3$ so that $Q(x, y) = 0$ is a pencil of lines passing through the base point where lines 1 and 3 intersect. For each choice of (a, b), the line and the curve intersect at the base point and at one other point (x, y). Thus

$$L_1 - a\,L_3 = 0 \tag{6.1a}$$

$$L_1\,L_2 - b\,L_3\,L_4 = 0 \tag{6.1b}$$

define a birational transformation between the (x, y) and the (a, b) coordinates. Thus we have a single Cremona transformation h between two planes, defined by rational functions such that $(a, b) = h(x, y)$, and h^{-1} defined by $(x, y) = h^{-1}(a, b)$. Define $\hat{g}_\alpha$ acting on the (a, b) coordinates by the replacement $\hat{g}_\alpha(a, b) = (\mathrm{e}^\alpha\,a, b)$. Then the composition $g_\alpha = h\,\hat{g}_\alpha\,h^{-1}$ is a one-parameter, connected Lie subgroup of $\mathcal{C}_2$ acting in the (x, y) plane.

A similar construction applies, with some loss of generality, for subgroups of $\mathcal{C}_n$ with higher n. For instance when $n = 3$ suppose $L_k(x, y, z) = 0$ for $k = 1, 2, \ldots, 6$ are planes. Define a mapping h and its inverse such that $(a, b, c) = h(x, y, z)$ and $(x, y, z) = h^{-1}(a, b, c)$ by solving

$$L_1 - a\, L_2 = 0 \tag{6.2a}$$

$$L_1\, L_3 - b\, L_2\, L_4 = 0 \tag{6.2b}$$

$$L_1\, L_5 - c\, L_2\, L_6 = 0 \tag{6.2c}$$

either for (a, b, c) or (x, y, z). Alternatively, a cubic subgroup of $\mathcal{C}_n$ is generated by h obtained from the set of equations

$$L_1 - a\, L_4 = 0 \tag{6.3a}$$

$$L_1\, L_2 - b\, L_4\, L_5 = 0 \tag{6.3b}$$

$$L_1\, L_2\, L_3 - c\, L_4\, L_5\, L_6 = 0. \tag{6.3c}$$

A single projective map can put the base points in standard locations. For $n = 2$ the base points can be placed at $(0, 0)$, $(0, \infty)$, $(\infty, 0)$ and at $(1, 1)$. Group flow can be chosen to be from $(0, 0)$ to $(1, 1)$. Two saddle points appear automatically at $(1, 0)$ and $(0, 1)$. Thus the group flow is characterized by a source, a sink, two saddle points and four invariant lines. The group invariant lines can be any combination of fixed or invariant lines of $\mathcal{S}$ or may belong to the pth preimage of some invariant set of $\mathcal{S}$. To see whether a given system $\mathcal{S}$ admits such a group, one proceeds as in the case of PGL(n) by examining fixed and invariant sets of the system. Since the group action $\hat{g}_\alpha$ is just dilation in b, the transformation h itself decouples the dynamical system.

7. Canonical forms and canonical variables

In this section canonical variables are tabulated for groups with elementary Lie algebras (see also [3]). Once a continuous symmetry has been found for $\mathcal{S}$ the next step is to reduce the order. To motivate the introduction of canonical variables, we illustrate this reduction for a simple example [25]. Consider a system with a two-variable recursion. If we represent components of the dynamical vectors by x and y to lighten the notation, then the recursion equations (2.1) are of the form

$$\begin{aligned} X &= f(x, y) \\ Y &= g(x, y). \end{aligned} \tag{7.1}$$

In this case only one symmetry is needed to decouple the system. Suppose a symmetry is generated by

$$\mathcal{L} = \xi^x(x, y)\, \frac{\partial}{\partial x} + \xi^y(x, y)\, \frac{\partial}{\partial y}. \tag{7.2}$$

Canonical variables (a, b) may be chosen such that

$$\mathcal{L} = \frac{\partial}{\partial a} \tag{7.3}$$

provided that two partial differential equations

$$\mathcal{L}\, a = 1 \tag{7.4a}$$

$$\mathcal{L}\, b = 0 \tag{7.4b}$$

can be solved to yield transformations

$$a = a(x, y) \qquad b = b(x, y) \tag{7.5a}$$

$$x = x(a, b) \qquad y = y(a, b). \tag{7.5b}$$

Expressed in terms of terms of (a, b) the general dynamical system becomes

$$\begin{aligned} A &= \hat{f}(a, b) \\ B &= \hat{g}(a, b) \end{aligned} \tag{7.6}$$

where the hats indicate functions f and g in equations (7.1) are transformed using equations (7.5*a*) and (7.5*b*). To see that a decoupled system results, return to the normal form of the symmetry generator. $\mathcal{L} = \partial/\partial a$ implies that the symmetry transformation may be written

$$a_\alpha = a + \alpha \qquad b_\alpha = b \tag{7.7}$$

where α is the group parameter. Since g_α and f must commute

$$\begin{aligned} A + \alpha &= \hat{f}(a + \alpha, b) \\ B &= \hat{g}(a + \alpha, b). \end{aligned} \tag{7.8}$$

However, because α is arbitrary one finds that

$$\begin{aligned} A &= a + \hat{f}(0, b) \\ B &= \hat{g}(0, b) \end{aligned} \tag{7.9}$$

so that in fact the function $\hat{g}(a, b)$ cannot depend on a, and $\hat{f}(a, b)$ must be linear in a. Thus the order of the system has reduced by one.

The obvious way to approach a system admitting an r-parameter group is to reduce first by using a one-parameter subgroup corresponding to a single generator. Then one would continue reducing step by step. The question is whether or not, after the pth reduction step, the reduced system still admits an $(r - p)$-parameter group. We proceed by organizing groups according to the structure of their Lie algebras [40].

First consider an n-dimensional system admitting an r-dimensional Abelian symmetry group $(r \leqslant n)$ such that

$$[\mathcal{L}_i, \mathcal{L}_j] = 0. \tag{7.10}$$

Also assume the symmetry generators are independent so that the tangent vectors at each point x span an r-dimensional space. We can augment the basis to include n generators in all, the first r of which are symmetries, any two of which commute, and such that the tangent vectors span n-dimensional space. In this simple case, each of the generators may be written in a normal form

$$\mathcal{L}_i = \frac{\partial}{\partial a^i}. \tag{7.11}$$

Equations (7.4*a*) and (7.4*b*) can be generalized to matrix form such that

$$\xi\, A_{,x} = 1 \tag{7.12}$$

where ξ is a matrix with (i, j)th entry ξ_i^j, $A_{,x}$ is the Jacobian matrix with $(A_{,x})_j^k = \partial a^k/\partial x^j$, and 1 is the identity matrix. With this notation, the transformation functions $\{a^i(x)\}$ are obtained from

$$A_{,x} = \xi^{-1} \tag{7.13}$$

and so

$$a^i(x) = \int \left[(\xi^{-1})^i_1 \, \mathrm{d}x^1 + (\xi^{-1})^i_2 \, \mathrm{d}x^2 + \cdots + (\xi^{-1})^i_n \, \mathrm{d}x^n\right]. \tag{7.14}$$

The form of the reduced system is

$$A^i = \begin{cases} a^i + \hat{f}^i(0, \ldots, a^{r+1}, \ldots, a^n) & \text{for } i \leqslant r \\ \hat{f}^i(0, \ldots, a^{r+1}, \ldots, a^n) & \text{for } i > r. \end{cases} \tag{7.15}$$

When the group is Abelian the generators commute in pairs, so the step-by-step reduction to equation (7.15) would seem to depend on whether or not the tangent vectors are independent. Consider for example an Abelian group with two generators $\mathcal{L}_1$ and $\mathcal{L}_2$ that commute but such that $\det(\xi)$ vanishes. In this case a change can be made to coordinates (a, b) where the generators become

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{7.16a}$$

$$\mathcal{L}_2 = b\,\frac{\partial}{\partial a}. \tag{7.16b}$$

The group transformations are

$$a_\alpha = a + \alpha \tag{7.17a}$$

$$a_\beta = a + \beta\, b. \tag{7.17b}$$

Following an argument like the one leading from equation (7.7) to equation (7.9), the fact that $\mathcal{L}_1$ and $\mathcal{L}_2$ must commute with the system map results in

$$\begin{aligned} A &= a + \hat{f}(0, b) \\ B &= b. \end{aligned} \tag{7.18}$$

The point is that since $\mathcal{L}_1$ and $\mathcal{L}_2$ are not independent it would appear that the order should reduce only by one. However, the solution for b is obviously constant, so another simplification occurs due to the second symmetry.

In a three-dimensional Abelian case for which

$$\det(\xi) = 0 \tag{7.19}$$

again one generator can be expressed in a normal form $\mathcal{L}_1 = \partial/\partial a$. The simplest case involving all three letters is

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{7.20a}$$

$$\mathcal{L}_2 = b\,\frac{\partial}{\partial a} \tag{7.20b}$$

$$\mathcal{L}_3 = c\,\frac{\partial}{\partial a} \tag{7.20c}$$

so that all three group generators act in the same one-dimensional subspace. A second case is

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{7.21a}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial b} \tag{7.21b}$$

$$\mathcal{L}_3 = v(c)\,\frac{\partial}{\partial a} + w(c)\,\frac{\partial}{\partial b}. \tag{7.21c}$$

For the first case in which the generators span a one-dimensional subspace, the group action must be

$$a_\alpha = a + \alpha \tag{7.22a}$$

$$a_\beta = a + \beta\, b \tag{7.22b}$$

$$a_\gamma = a + \gamma\, c \tag{7.22c}$$

where the notation follows equations (7.20*a*)-(7.20*c*) and no formulae are given for b or c because these coordinates are unchanged. Since the subgroup corresponding to each parameter α, β, and γ must commute with the system mapping, the recursions reduce completely to

$$\begin{aligned} A &= a + \hat{f}(0, b, c) \\ B &= b \\ C &= c. \end{aligned} \tag{7.23}$$

Now consider the second case represented by equations (7.21*a*)-(7.21*b*) in which the generators act in a two-dimensional subspace. The group action must be

$$a_\alpha = a + \alpha \tag{7.24a}$$

$$b_\beta = b + \beta \tag{7.24b}$$

$$a_\gamma = a + \gamma\, v(c) \qquad b_\gamma = b + \gamma\, w(c) \tag{7.24c}$$

where again no formulae are given for the unchanged coordinates. The reduced recursions are

$$\begin{aligned} A &= a + \hat{f}(0, 0, c) \\ B &= b + \hat{g}(0, 0, c) \\ C &= c. \end{aligned} \tag{7.25}$$

Thus, in each case the order is reduced and the system decouples completely.

To illustrate the case of independent generators that do not commute in pairs, i.e. $[\mathcal{L}_i, \mathcal{L}_j]$ is not zero for every i and j, we begin with the example of a two-variable dynamical system. Two generators $\mathcal{L}_1$ and $\mathcal{L}_2$, can be found such that

$$[\mathcal{L}_1, \mathcal{L}_2] = \mathcal{L}_1. \tag{7.26}$$

This is the most general non-Abelian case with two independent generators [3, 40]. In terms of a and b a normal form is

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{7.27a}$$

$$\mathcal{L}_2 = a\,\frac{\partial}{\partial a} + b\,\frac{\partial}{\partial b} \tag{7.27b}$$

representing simple translation and dilation. The transformation associated with $\mathcal{L}_1$ is

$$a_\alpha = a + \alpha. \tag{7.28}$$

The second transformation acts on both a and b such that

$$a_\beta = \mathrm{e}^\beta\, a \qquad b_\beta = \mathrm{e}^\beta\, b. \tag{7.29}$$

These symmetries must commute with the system mapping. The application of the two transformations equation (7.28) and (7.29) in either order results in a two-parameter transformation of the same form but with a slightly different parametrization. The reduced system in (a, b) becomes

$$\begin{aligned} A &= a + \eta_1 b \\ B &= \eta_2 b \end{aligned} \tag{7.30}$$

where η_1 and η_2 are constants. The reduced equations (7.30) solve trivially.

Next consider the three dimensional Lie algebra with commutators

$$\begin{aligned} [\mathcal{L}_1, \mathcal{L}_2] &= 0 \\ [\mathcal{L}_1, \mathcal{L}_3] &= 0 \\ [\mathcal{L}_2, \mathcal{L}_3] &= \mathcal{L}_1 \end{aligned} \tag{7.31}$$

where it is assumed that $\det(\xi)$ does not vanish. A normal form is

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{7.32a}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial b} \tag{7.32b}$$

$$\mathcal{L}_3 = b\,\frac{\partial}{\partial a} + c\,\frac{\partial}{\partial c}. \tag{7.32c}$$

The associated transformations in (a, b, c) are

$$a_\alpha = a + \alpha \tag{7.33a}$$

$$b_\beta = b + \beta \tag{7.33b}$$

$$a_\gamma = a + \gamma\, b \qquad c_\gamma = \mathrm{e}^\gamma\, c. \tag{7.33c}$$

The reduced system is

$$\begin{aligned} A &= a + \eta_1 \\ B &= b \\ C &= \eta_2\, c \end{aligned} \tag{7.34}$$

where η_1 and η_2 are constants.

Finally, consider non-commuting generators for which the tangent vectors are not independent, so that $\det(\xi)$ vanishes. To demonstrate this, we focus again on the case of two generators. Assume the commutator between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$[\mathcal{L}_1, \mathcal{L}_2] = \mathcal{L}_1. \tag{7.35}$$

A normal form in this case is

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{7.36a}$$

$$\mathcal{L}_2 = a\,\frac{\partial}{\partial a}. \tag{7.36b}$$

In normal coordinates the reduction of order results from the corresponding finite transformations

$$a_\alpha = a + \alpha \tag{7.37a}$$

$$a_\beta = \mathrm{e}^\beta\, a. \tag{7.37b}$$

This leads directly to

$$\begin{aligned} A &= a \\ B &= \hat{g}(0, b) \end{aligned} \tag{7.38}$$

so that the system decouples using either symmetry, but even using both together does not result in a complete solution, since the recursion for b is quite general.

At this time we know only the following regarding a general criterion to guarantee that a particular $\mathcal{G}$ with r generators will reduce the order of a system $\mathcal{S}$ with which it commutes by exactly r. The algebra defined in each example in this section is solvable in the sense that the limit of the sequence of derived algebras is the algebra containing only zero [40]. Examples given by equations (7.27), (7.32) and (7.36) are non-Abelian. Only in the non-Abelian case with dependent tangent vectors, $\det(\xi) = 0$, are we unable to reduce the order of the system by r, the number of group parameters.

8. Examples

The purpose of this section is to illustrate the use of tools developed in sections section 4 through 7. Most of the examples of dynamical systems are drawn from other papers in order to demonstrate the relevance of the group theoretic methods to actual physical problems. Only a cursory explanation of the original problem is given for each example. We have attempted to suggest physical meanings for the variables. The interested reader is invited to consult the references cited for a full definition and further information. Our main concern will be with reducing the order of the dynamical systems in each case.

Adler [44] considers the problem of Stokes flow on an anisotropic Sierpiński gasket, each triangular unit cell of which has equal resistance on two edges and different resistance on the third. Recursions for the elements of a transfer matrix relating pressure to flow rate are

$$\begin{aligned} X &= \frac{x\,(7\,x + 4\,y)\,(x - y)}{2\,(2\,x - y)\,(3\,x + y)} \\ Y &= \frac{(x - y)\,(x^2 - 4\,x\,y - 2\,y^2)}{2\,(2\,x - y)\,(3\,x + y)} \end{aligned} \tag{8.1}$$

where y is the conductance along the odd edge and x is the negative of the sum of the conductance along one of the equal edges and conductance along the odd edge. Since the recursions are homogeneous, they admit a dilation symmetry. Thus

$$\mathcal{L} = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \tag{8.2}$$

which leads by way of equations (7.2) through (7.4*b*) to

$$a = \ln(1/x) \qquad b = y/x \tag{8.3}$$

for canonical variables. The recursion equations (8.1) become

$$\begin{aligned} A &= a + \ln\left[\frac{2\,(b-2)\,(b+3)}{(b-1)\,(4\,b+7)}\right] \\ B &= -\,\frac{2\,b^2 + 4\,b - 1}{4\,b + 7}. \end{aligned} \tag{8.4}$$

Recognizing homogeneity of equations (8.1), Adler has used canonical variables to decouple them and study scaling of the conductance. Adrover *et al* [45] study scaling cross-over in a related model for diffusion in anisotropic media and use dilation symmetry to decouple similar recursions. Giona [30, 28] observed that the dilation symmetry in circuit models of this sort comes form the fact that scaling the resistance of each link by λ merely scales the total resistance by λ. This is related to current conservation.

Suppose one did not notice the dilation symmetry of equations (8.1). The analysis would proceed as follows. The only fixed points are $(0, 0)$ and (∞, ∞). The points at ∞ map into one another, so the line at ∞ is invariant under $\mathcal{S}$. This allows the line at ∞ to be fixed by $\mathcal{G}$. The system can admit the PGL(2) subgroup belonging to the first Jordan form in equation (5.1) with two equal eigenvalues. The inverse image of $(0, 0)$ contains the line $x - y = 0$. Successive inverse images contain lines forming a pencil through $(0, 0)$. The group flow is radial. Therefore the system may admit uniform dilation.

Considering vibrations of the Vicsek lattice [46], Jayanthi and Wu [47] develop parametric recursions for determining the frequency ω from the equation of motion for transverse vibration of the lattice with free-end boundary conditions. The four parameters α, β, $\tilde{\beta}$ and κ of Jayanthi and Wu representing renormalized squared frequencies and coupling constants are taken as x, y, z and k for our purposes. The recursions are

$$\begin{aligned}
X &= x + \frac{4\,(4-z)\,k^2}{(4-z)^2 - k^2} \\
Y &= x + \frac{3\,k^2}{4-y} + \frac{(4-z)\,k^2}{(4-z)^2 - k^2} \\
Z &= x + \frac{2\,k^2}{4-y} + \frac{2\,(4-z)\,k^2}{(4-z)^2 - k^2} \\
K &= \frac{k^3}{(4-z)^2 - k^2}.
\end{aligned} \tag{8.5}$$

As in the previous example, an analysis of the fixed and invariant sets reveals a dilation symmetry. However, the homogeneity is obscured by a shift in the centre of dilation to the point $(4, 4, 4, 0)$. This model admits a subgroup of PGL(4). From this symmetry we find the canonical variables

$$\begin{aligned}
a &= (x-4)/k \qquad & b &= (y-4)/k \\
c &= (z-4)/k \qquad & d &= \ln k.
\end{aligned} \tag{8.6}$$

In addition, the recursion equations in (a, b, c, d) are such that

$$4\,A + 8\,B - 12\,C = 0. \tag{8.7}$$

Thus the number of necessary variables is reduced by one. Therefore, the number of recursions is reduced again by one. Solving for a in terms of b and c, the final form of the recursions is

$$\begin{aligned}
B &= -\frac{2\,b^2\,c^2 - 3\,b\,c^3 - 2\,b^2 + 3\,c^2 + 4\,b\,c - 3}{b} \\
C &= -\frac{2\,b^2\,c^2 - 3\,b\,c^3 - 2\,b^2 + 2\,c^2 + 5\,b\,c - 2}{b} \\
D &= d + \ln(c^2 - 1).
\end{aligned} \tag{8.8}$$

The above result does not admit a further subgroup of PGL(n) or a Cremona subgroup of the sort developed in section 7 for the following reasons. Consider the (b, c) recursions. One finds eight fixed points on two parallel lines, four on each line. This immediately excludes the PGL(2) subgroups. The Cremona subgroups are eliminated because both the total number of fixed points and the number per line exceeds the maximum allowed. In fact M Schwalm *et al* showed that the vibrational problem for the Vicsek lattice with free-end boundary conditions can be renormalized in such a way that the recursions decouple completely [48].

Hood and Southern [49] consider Schrödinger eigenstates of an anisotropic fractal lattice. The model consists of a tight-binding Hamiltonian matrix H on a Sierpiński lattice with strong and weak bonds. The strong bonds form a continuous self-avoiding walk from one corner to another. They derive decimation recursion relations for variables which we shall call (x, y, z) related to the energy and bond strength. The recursions are

$$\begin{aligned} X &= \frac{2x^2yz + x^3 + 2xyz + yz}{1 - 3x^2 - 4yz - 6xyz} \\ Y &= \frac{(x+1)\,y\,(2yz + x)}{1 - 3x^2 - 4yz - 6xyz} \\ Z &= \frac{z\,(2yz + x)}{1 - x - 6yz}. \end{aligned} \tag{8.9}$$

We find two isolated fixed points $(-\frac{1}{2}, 0, 0)$ and $(0, 0, 0)$ and two fixed lines $(x = \frac{1}{2},\ y = 0)$ and $(x = \frac{1}{4},\ y = \frac{1}{16}z)$. These lines must be group trajectories. Taking further inverse images, we find more group trajectories of the form $(y = b/z,\ x = c)$. Each trajectory is mapped by the system onto another of the same form but for different b and c. Thus, the group flow in the (y, z) subspace is $(e^{\alpha}\,y,\ e^{-\alpha}\,z)$ so that

$$\mathcal{L} = y\,\frac{\partial}{\partial y} - z\,\frac{\partial}{\partial z}. \tag{8.10}$$

In convenient canonical variables

$$a = \ln y \qquad b = yz \qquad c = x \tag{8.11}$$

the system reduces to

$$\begin{aligned} A &= a + \ln\left[\frac{(c+1)\,(2b+c)}{1 - 4b - 3c^2 - 6bc}\right] \\ B &= \frac{(b\,(c+1)\,(2b+c)}{(6b + c - 1)\,(3c^2 + 6bc + 4b - 1)} \\ C &= \frac{c^3 + 2bc^2 + 2bc + b}{1 - 4b - 3c^2 - 6bc}. \end{aligned} \tag{8.12}$$

Once again, the system equations (8.9) admit a subgroup of PGL(3) which results in reduction of order. Hence the model of Hood and Southern reduces to a dynamical system in two variables.

The recursions for b and c do not admit any of the subgroups discussed above because the number of isolated fixed points and crossings of their successive inverse images forces the number of group fixed points to grow beyond the limits imposed by the properties of PGL(2) or $\mathcal{C}_2$.

One way to treat dynamical models like diffusion, vibrations or the Schrödinger equation that contain a Laplacian operator is to make space discrete. The difference Laplacian involves the adjacency matrix H for a graph representing a quadrature grid. Physical quantities of interest are computed efficiently from Green functions. If Laplace or Fourier transforms are taken from time to some complex transform parameter E, the Green functions or transfer functions are entries of $G(E) = (E - H)^{-1}$.

As an example involving Green functions, consider the modified rectangle lattice of Dhar [50]. Reese *et al* [51] derived renormalization recursion relations for a small set of Schrödinger propagators on the modified rectangle. If i, j and k represent sites on three consecutive corners of the rectangle, then $x = G_{i,i}$, $y = G_{i,j}$, $u = G_{j,k}$ and $v = G_{k,i}$. The recursions can be simplified using point symmetrized variables [52] (p, q, r, s) such that

$$\begin{aligned} p &= x + y + u + v \qquad & q &= x - y + u - v \\ r &= x + y - u - v \qquad & s &= x - y - u + v. \end{aligned} \tag{8.13}$$

The recursions in (p, q, r, s) are

$$\begin{aligned} P &= \frac{p + q - 2\,p\,q}{2 - p - q} \\ Q &= \frac{r + s - 2\,r\,s}{2 - r - s} \\ R &= \frac{p + q + 2\,p\,q}{2 + p + q} \\ S &= \frac{r + s + 2\,r\,s}{2 + r + s}. \end{aligned} \tag{8.14}$$

One finds that hyperplanes $p = 1$ and $s = -1$ are invariant. The preimage of $p = 1$ contains $r = 1$ and the preimage of $r = 1$ contains $q = 1$ and $s = 1$. The preimage of $s = -1$ contains $q = -1$, the preimage of which is $r = -1$. We find group invariants and fixed points at the intersections of the hyperplanes $p = \pm 1$, $q = \pm 1$, $r = \pm 1$ and $s = \pm 1$. A Cremona transformation h defined using

$$(1 + p) - a\,(1 - p) = 0 \tag{8.15a}$$

$$(1 - p)\,(1 + q) - b\,(1 + p)\,(1 - q) = 0 \tag{8.15b}$$

$$(1 - p)\,(1 + r) - c\,(1 + p)\,(1 - r) = 0 \tag{8.15c}$$

$$(1 - p)\,(1 + s) - d\,(1 + p)\,(1 - s) = 0 \tag{8.15d}$$

leads via equations (6.1*a*) and (6.1*b*) to the $\mathcal{C}_4$ subgroup generated by

$$\mathcal{L} = \tfrac{1}{2}\left[(p^2 - 1)\,\frac{\partial}{\partial p} + (q^2 - 1)\,\frac{\partial}{\partial q} + (r^2 - 1)\,\frac{\partial}{\partial r} + (s^2 - 1)\,\frac{\partial}{\partial s}\right]. \tag{8.16}$$

The introduction of canonical variables (a, b, c, d) gives the reduced system

$$\begin{aligned} A &= \frac{2\,a\,b}{b + 1} \\ B &= \frac{c\,d\,(b + 1)}{b\,(c + d)} \\ C &= \frac{(b + 1)^2}{4\,b} \\ D &= \frac{(b + 1)\,(c + d)}{4\,b} \end{aligned} \tag{8.17}$$

where the decoupling of a may be clearer using the proper canonical variable $\ln a$, but the relation between equations (6.2*a*)–(6.2*c*) and equations (8.15*a*)–(8.15*d*) is more clear when the logarithm is not used.

As a final example consider the recursions obtained by W Schwalm *et al* [28] for Schrödinger Green functions of a fourfold coordinated Sierpiński lattice with the corner sites removed. For the symmetry adapted propagators (p, q, r, s, t) defined in [28], the recursions are

$$\begin{aligned}
P &= p + \frac{t^2(1-4\,p)}{(1-2\,t-4\,p)\,(1+t-4\,p)} \\
Q &= q + \frac{(t-2\,r)^2}{1+t-4\,p} \\
R &= \frac{t\,(2\,t^2+r-2\,t\,r-4\,p\,r)}{(1-2\,t-4\,p)\,(1+t-4\,p)} \\
S &= -\frac{(t-2\,r)\,r}{1+t-4\,p} \\
T &= \frac{t^2\,(1+2\,t-4\,p)}{(1-2\,t-4\,p)\,(1+t-4\,p)}.
\end{aligned} \tag{8.18}$$

Constructing fixed and invariant sets and proceeding as in section 5 one finds that this system admits the PGL(5) symmetry subgroup generated by

$$\mathcal{L}_1 = \frac{\partial}{\partial q} \tag{8.19a}$$

$$\mathcal{L}_2 = \left(p - \frac{1}{4}\right)\frac{\partial}{\partial p} + q\,\frac{\partial}{\partial q} + r\,\frac{\partial}{\partial r} + s\,\frac{\partial}{\partial s} + t\,\frac{\partial}{\partial t} \tag{8.19b}$$

$$\mathcal{L}_3 = \left(p - \frac{1}{4}\right)\frac{\partial}{\partial p} + \left(\frac{r}{2} + \frac{t}{4}\right)\frac{\partial}{\partial r} + \left(\frac{r}{2} - \frac{t}{4}\right)\frac{\partial}{\partial s} + t\,\frac{\partial}{\partial t}. \tag{8.19c}$$

These generators commute except for $[\mathcal{L}_1, \mathcal{L}_2] = \mathcal{L}_1$, so a normal form is

$$\mathcal{L}_1 = \frac{\partial}{\partial a} \tag{8.20a}$$

$$\mathcal{L}_2 = a\,\frac{\partial}{\partial a} + b\,\frac{\partial}{\partial b} \tag{8.20b}$$

$$\mathcal{L}_3 = \frac{\partial}{\partial c}. \tag{8.20c}$$

The canonical variables

$$\begin{aligned}
a &= q \\
b &= t\left(1 - \frac{2\,r}{t}\right)^2 \\
c &= \ln\left(1 - \frac{2\,r}{t}\right) \\
d &= \frac{(t-2\,r)^2}{t\,(t-2\,r+2\,s)} \\
f &= \frac{1-4\,p}{t}
\end{aligned} \tag{8.21}$$

obtained from the normal form for the generators, lead to the recursions

$$
\begin{aligned}
A &= a + \frac{b}{f+1} \\
B &= \frac{b\,(f-2)}{(f+1)\,(f+2)} \\
C &= c + \ln\left(\frac{f-2}{f+2}\right) \\
D &= \frac{f-2}{f+2} \\
F &= f\,(f-3).
\end{aligned}
\tag{8.22}
$$

The system is essentially decoupled.

9. Conclusion

The procedures presented above reduce the order of many discrete dynamical systems arising from renormalization of lattice problems. Several of these appear as examples in section 8. In each case known to us where the method succeeds, the system admits a subgroup of PGL(n) or a quadratic Lie subgroup of $\mathcal{C}_n$ of the form discussed in section 6. The central problem of deducing whether or not a given system admits a group of either of these types is addressed in sections 5 and 6.

The relation discussed in section 4 between the invariant sets of a group and of the systems it admits has shown to be a key element, both in characterizing systems admitted by a given type of group and in using the group to reduce the order of a system. Thus the groups dealt with in sections 5 and 6 are classified according to their fixed and invariant sets. The method of constructing canonical group coordinates, once the symmetry group is known, and the method of reduction of order is developed systematically in section 7.

It is of at least equal importance that one can often show conclusively by generating invariant sets that a given system does not admit a symmetry of either of the types we have examined. The utility of this should not be underestimated.

Use has been made of two group classifications, a global one in terms of fixed and invariant sets in the space where the group transformations operate and a local one in terms of Lie algebras. The first one is made modulo fixed transformations of PGL(n) or $\mathcal{C}_n$ that put the fixed and invariant sets of $\mathcal{G}$ in a standard configuration. The second classification is made modulo the linear transformations in the space of generators which corresponds to reparametrizing the group. One is then left with a finite set of normal forms. Results presented above suggest that continuing a systematic program of classification and reduction of this sort may lead to results quite useful for analyzing general discrete systems. On the other hand, at the present time even the general $\mathcal{C}_n$ is not fully characterized for $n \geqslant 3$, so the synthetic method cannot produce fully general results.

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